OPTIMIZATION OF DETERMINISTIC POPULATION DYNAMICS MODELS

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Abstract

The aim of this contribution is to apply methods from optimal control theory to the mathematical modeling of biological pest control. We formulate a pest control strategy for nonlinear Kolmogorov system of *n* interacting populations by introducing natural enemies as a control function. The sufficient conditions for existence of an optimal feedback control function are based on the fact, that the steady-state solution of the Hamilton-Jacobi-Bellman partial differential equation is a Lyapunov function guaranteeing stability and optimality We apply those general results to the Lotka-Volterra system with a logistic rate of increase of the prey population and Holling's second type functional response of the predator population, to illustrate biological control of pest mite in stored grain *Acarus siro* by predatory mite *Cheyletus eruditus*.

Keywords: Kolmogorov system, optimal control theory, Lotka-Volterra system, pest control strategy.

JEL classification: J110, J180, J19

1. Introduction

Optimal control theory acquires an increased application in both theoretical and applied ecology and epidemiology. It is a main method of adaptive resource management, sustainable ecosystem management ([13]), optimal harvesting and foraging theory ([1], [8], [12]), native-invasive population dynamics ([6], [3]) and pest management programs ([14]).

Most of pest control methods are based on chemical insecticides. Chemical controls are inexpensive to use and very effective but with high environmental cost. There appears a tendency to use natural enemies to suppress pest population and thus making it less damaging. This approach is called biological control (BC). Role of BC is to stabilize the density of the pest population in the level of non-economic and non-ecologic damage by predators, parasites, parasitoids or pathogens. BC sometimes includes genetic manipulations to increase the resistance of organisms such as sterilization and disturbance of mate finding ability ([4]).

The theoretical and experimental studies show, that successful natural enemy must have some qualities: it is specific for the pest population, it has high intrinsic growth rate, high search capability, synchronous dynamic and aggregates in areas with a high density of pest population. There are three types of BC: (1) importing and introducing natural enemies of the original geographic area, (2) augmentation of a large population of natural enemies for immediate effect or periodic augmentation ofa small population of natural enemies, (3) conservation of environment in order to preserve existing natural enemies.

In this paper we focus on the use of optimal control theory to the mathematical modeling of BC through the non-recurring augmentation of natural enemies. In general BC is modeled as predator-prey or host-parasitoid type systems. We organized this paper as follows. In Section 2 we present the optimal control problem of the nonlinear Kolmogorov system. In Section 3 we apply this approach to control the predator-prey system and finally conclusion and references are given.

2. Formulation of the control problem

Optimal control theory is one of several applications and extensions of the calculus of variations ([5]). It deals with finding an admissible control function that minimizes the performance measure functional with differential equation constraints. It is known, that the nonlinear optimal control problem (nonquadratic functional with nonlinear differential equations) can be reduced to the Hamilton-Jacobi-Bellman nonlinear partial differential equation ([7]). There are many problems in its solution except the case of linear regulator problem (quadratic functional with linear differential equations), where the Hamilton-Jacobi-Bellman partial equation is reduced to the Riccati system of nonlinear ordinary differential equations and it can be shown, that the solution is a Lyapunov function.

Bernstein ([2]) rewied a framework for optimal nonlinear problems involving nonquadratic functionals that is analogous to linear-quadratic theory. He investigated time-invariant systems on the infinite interval. In such case the steady state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function of the nonlinear system, guaranteeing stability and optimality.

We formulate a pest control strategy for Kolmogorov model of n interacting species

$$\dot{x}_i = x_i\, f_i(x_1, x_2, \ldots, x_n) = g_i(x_1, x_2, \ldots, x_n) \tag{1}$$

or in vector form

$$\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x}), \tag{2}$$

where χ_i is density of i-th species at the time t, $f_i(\chi_1,\chi_2,\ldots,\chi_n)$ is nonlinear continuous function for $i = 1,\ldots,n$ and it expresses growth rate of the i-th species depending on densities of all of the species.

Suppose, that the right-hand side of (2) can be divided into linear and nonlinear parts

$$\dot{\mathbf{x}} = \mathrm{A}\mathbf{x} + \mathbf{h}(\mathbf{x}), \tag{3}$$

where $\mathrm{A} \in \mathbb{R}^{n\times n}$ is a constant matrix, $\mathbf{h}(\chi)$ is a vector of nonlinear functions. We introduce into the system (3) natural enemies represented by control vector $\mathbf{u} \in \mathbb{R}^m$ with constant matrix $\mathrm{B} \in \mathbb{R}^{n\times m}$:

$$\dot{\mathbf{x}} = \mathrm{A}\mathbf{x} + \mathbf{h}(\mathbf{x}) + \mathrm{B}\mathbf{u}. \tag{4}$$

The aim of this control is to move the system to the desired steady state $\mathbf{x}^* = [x_1^*, \ldots, x_n^*]^\top$, in which the pest density is stable without causing damages and natural enemy density is stabilized at the level, that will allow further control. The desired steady state satisfies the following system

$$\mathbf{0} = \mathrm{A}\mathbf{x}^* + \mathbf{h}(\mathbf{x}^*) + \mathrm{B}\mathbf{u}^*.$$

But in general, the desired steady state can be unstable. To avoid this obstacle, the control strategy must be sum of two control vectors $\mathbf{u} = \mathbf{u}^* + \tilde{\mathbf{u}}$, where $\tilde{\mathbf{u}}$ ensures asymptotical stability.

Define new variables

$$\mathbf{y} = \mathbf{x} - \mathbf{x}^*, \quad \tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}^*. \tag{5}$$

Substituting (5) into (4) we get the "error" system:

$$\begin{aligned} \dot{\mathbf{y}} &= \mathrm{A}(\mathbf{y} + \mathbf{x}^*) + \mathbf{h}(\mathbf{y} + \mathbf{x}^*) + \mathrm{B}(\mathbf{u}^* + \tilde{\mathbf{u}}) = \\ &= \mathrm{A}\mathbf{y} + \mathbf{h}(\mathbf{y} + \mathbf{x}^*) + \mathrm{B}\tilde{\mathbf{u}} + \mathrm{A}\mathbf{x}^* + \mathrm{B}\mathbf{u}^* + \mathbf{h}(\mathbf{x}^*) - \mathbf{h}(\mathbf{x}^*) = \\ &= \mathrm{A}\mathbf{y} + \mathbf{h}(\mathbf{y} + \mathbf{x}^*) - \mathbf{h}(\mathbf{x}^*) + \mathrm{B}\tilde{\mathbf{u}} = \\ &= \mathrm{A}\mathbf{y} + \mathbf{q}(\mathbf{y}) + \mathrm{B}\tilde{\mathbf{u}}, \end{aligned} \tag{6}$$

where $\mathbf{q}(\mathbf{y}) = \mathbf{h}(\mathbf{y} + \mathbf{x}^*) - \mathbf{h}(\mathbf{x}^*)$.

Based on the results of ([2]) we prove the next theorem:

Theorem 1. Consider general nonlinear system with time-dependent $\mathrm{A}(t) \in \mathbb{R}^{n\times n}$

$$\dot{\mathbf{y}} = \mathrm{A}(t)\mathbf{y} + \mathbf{q}(\mathbf{y}) + \mathrm{B}\tilde{\mathbf{u}}, \qquad \mathbf{y}(0) = \mathbf{y}_0,\ \mathbf{q}(\mathbf{0}) = \mathbf{0}. \tag{7}$$

If there exist symmetric positive definite matrix $\mathbf{Q}(t)$ and positive definite matrix $\mathbf{R}(t)$, such that

$$l(\mathbf{y}) = \mathbf{y}^\top \mathrm{Q}\mathbf{y} - \mathbf{q}(\mathbf{y})^\top \mathrm{P}\mathbf{y} - \mathbf{y}^\top \mathrm{P}\mathbf{q}(\mathbf{y})$$

is positive definite function. Then the linear control function

$$\tilde{\mathbf{u}} = -\mathrm{R}^{-1}\mathrm{B}^{\top}\mathrm{P}\mathbf{y}\,, \tag{8}$$

that moves the system (7) from initial state to the desired final state $\mathbf{y}(t_f) = \mathbf{0}$ minimizing functional

$$\mathcal{J} = \int_0^{t_f} \mathrm{L}(\mathbf{y}(t), \tilde{\mathbf{u}}(t))\mathrm{d}t = \int_0^{t_f} \Big(l(\mathbf{y}(t)) + \tilde{\mathbf{u}}(t)^{\top}\mathrm{R}\tilde{\mathbf{u}}(t)\Big)\,\mathrm{d}t\,, \tag{9}$$

is optimal; the matrix **P**(t) is symmetric positive definite solution of Riccati equation

$$\dot{\mathrm{P}} + \mathrm{PA} + \mathrm{A}^{\top}\mathrm{P} - \mathrm{PBR}^{-1}\mathrm{B}^{\top}\mathrm{P} + \mathrm{Q} = \mathbf{0}\,. \tag{10}$$

Proof. According to the Theorem 3.1 and Remark 4.1 of ([2]) we know, that if the minimum of (9) exists and V is a smooth function of the initial conditions, then it satisfies the following Hamilton-Jacobi-Bellman equation:

$$\min_{\tilde{\mathbf{u}}}\big(\mathrm{L}(\mathbf{y}, \tilde{\mathbf{u}}) + V'(\mathbf{y})(\mathrm{A}\mathbf{y} + \mathbf{q}(\mathbf{y}) + \mathrm{B}\tilde{\mathbf{u}})\big) = \min_{\tilde{\mathbf{u}}}\left(l(\mathbf{y}) + \tilde{\mathbf{u}}^{\top}\mathrm{R}\tilde{\mathbf{u}} + \frac{\mathrm{d}V(\mathbf{y})}{\mathrm{d}t}\right) = 0 \tag{11}$$

where $'$ denotes the Fréchet derivative. Let us consider function

$$V(\mathbf{y}) = \mathbf{y}^{\top}\mathrm{P}(t)\mathbf{y}\,, \tag{12}$$

where **P**(t) satisfies (10). Then the derivative on the optimal trajectory with (8) according to (7) is

$$\begin{aligned}\frac{\mathrm{d}V(\mathbf{y})}{\mathrm{d}t} &= \dot{\mathbf{y}}^{\top}\mathrm{P}\mathbf{y} + \mathbf{y}^{\top}\dot{\mathrm{P}}\mathbf{y} + \mathbf{y}^{\top}\mathrm{P}\dot{\mathbf{y}} = \\ &= \mathbf{q}(\mathbf{y})^{\top}\mathrm{P}\mathbf{y} - \mathbf{y}^{\top}\mathrm{P}^{\top}\mathrm{B}(\mathrm{R}^{-1})^{\top}\mathrm{B}^{\top}\mathrm{P}\mathbf{y} - \mathbf{y}^{\top}\mathrm{Q}\mathbf{y} + \mathbf{y}^{\top}\mathrm{P}\mathbf{q}(\mathbf{y})\,.\end{aligned} \tag{13}$$

Substituting (13) into (11) we obtain

$$\mathbf{q}(\mathbf{y})^{\top}\mathrm{P}\mathbf{y} - \mathbf{y}^{\top}\mathrm{P}^{\top}\mathrm{B}(\mathrm{R}^{-1})^{\top}\mathrm{B}^{\top}\mathrm{P}\mathbf{y} - \mathbf{y}^{\top}\mathrm{Q}\mathbf{y} + \mathbf{y}^{\top}\mathrm{P}\mathbf{q}(\mathbf{y}) + l(\mathbf{y}) + \mathbf{y}^{\top}\mathrm{P}^{\top}\mathrm{B}(\mathrm{R}^{-1})^{\top}\mathrm{B}^{\top}\mathrm{P}\mathbf{y} = 0$$

so from that

$$l(\mathbf{y}) = \mathbf{y}^{\top}\mathrm{Q}\mathbf{y} - \mathbf{q}(\mathbf{y})^{\top}\mathrm{P}\mathbf{y} - \mathbf{y}^{\top}\mathrm{P}\mathbf{q}(\mathbf{y})\,.$$

Then $\dot{V}$ equals to $\dot{V} = -l(\mathbf{y}) - \tilde{\mathbf{u}}^{\top}\mathrm{R}\tilde{\mathbf{u}}$ and it is negative definite for both $l(\mathbf{y})$, R positive definite. Thus function V(**y**) is Lyapunov and then the system is locally asymptotically stable.

Remark 1. Theorem 1 holds also for the systems with constant matrix A. In such case $t_f = \infty$, matrices Q, R are constant and constant matrix P is symmetric positive definite solution of algebraic Riccati equation

$$\mathrm{PA} + \mathrm{A}^{\top}\mathrm{P} - \mathrm{PBR}^{-1}\mathrm{B}^{\top}\mathrm{P} + \mathrm{Q} = \mathbf{0}\,.$$

3. Application to the biological control of *Acarus siro*

The mite *Acarus siro* is one of the most important pests of stored products. It infests cereals, oilseeds, cheese, grain. Biological control was already developed 40 years ago using a predatory mite *Cheyletus eruditus* ([10]). To describe Acarus-Cheyletus dynamics mathematically we use the predator-prey model with Holling II type functional response of predator population and logistic growth of prey population ([9]):

$$\dot{x} = rx\left(1 - \frac{x}{K}\right) - \frac{axy}{1 + aTx}\,, \tag{14}$$

$$\dot{y} = c\frac{axy}{1 + aTx} - dy\,. \tag{15}$$

Here, $\chi = \chi(t)$ and $y = y(t)$ denote a time dependent abundance of prey *Acarus* and predator *Cheyletus* populations, respectively, the parameters r, K denote the prey intrinsic growth rate, carrying capacity of the environment for the prey population and the parameters d, c, α, T denote the predator mortality rate, conversion efficiency, predation rate (also called capture efficiency, search rate), handling time including time required for chasing, killing, eating and digesting the prey.

The standard qualitative analysis states that the system (14), (15) has one stable interior equilibrium. This fact is also illustrated by simulations performed in R ([11]) for given parameters r = 0.4, K = 500, d = *0.08*, c = 0.8, T = 0.5, α = 0.001 ([10]) with initial state [100;20], [300;2] respectively, in the Figure 1.

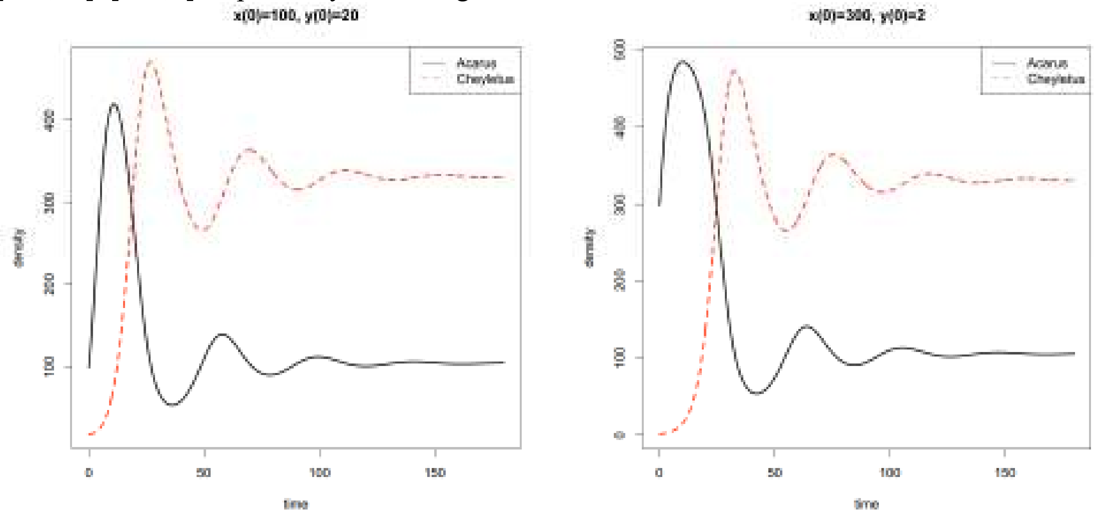

Figure 1. Density variations of the Acarus-Cheyletus model (14), (15) without control strategy

The aim is to control the dynamics by massive augmentation of the natural enemy *Cheyletus* to move the system to the desired steady-state χ^*, y^*, where the pest abundance χ^* should be equal and below the pest damage level. The controlled system (14), (15) is of the following form:

$$\dot{x} = rx\left(1-\frac{x}{K}\right)-\frac{axy}{1+aTx}, \quad (16)$$

$$\dot{y} = c\,\frac{axy}{1+aTx}-dy+u, \quad (17)$$

and the next holds

$$0 = rx^*\left(1-\frac{x^*}{K}\right)-\frac{ax^*y^*}{1+aTx^*}, \quad (18)$$

$$0 = c\,\frac{ax^*y^*}{1+aTx^*}-dy^*+u^*. \quad (19)$$

Since χ^* is given by the known values causing damage, from (18) we get

$$y^* = \frac{r}{a}(1+ax^*T)\left(1-\frac{x^*}{K}\right)$$

and from (19) we get

$$u^* = r\left(1-\frac{x^*}{K}\right)\left(\frac{d}{a}+dx^*T-cx^*\right).$$

Now substituting new variables $\tilde{u}=u-u^*$, $z=x-x^*$, $v=y-y^*$ into (16), (17) we construct an error system:

$$\dot{z} = rz\left(1-2\frac{x^*}{K}\right)-\frac{rz^2}{K}+rx^*-\frac{r(x^*)^2}{K}-v\left(\frac{1}{T}-\frac{1}{T(azT+ax^*T+1)}\right)-\frac{y^*(az+ax^*)}{azT+ax^*T+1},$$

$$\dot{v} = -dv-dy^*+v\left(\frac{c}{T}-\frac{c}{T(azT+ax^*T+1)}\right)+\frac{cy^*(az+ax^*)}{azT+ax^*T+1}+u^*+\tilde{u}.$$

The linear part of the error system can be represented by constant matrix

$$\mathrm{A} = \begin{pmatrix} r\left(1-2\frac{x^*}{K}\right) & -\frac{1}{T} \\ 0 & -d+\frac{c}{T} \end{pmatrix} \quad (20)$$

Then the control matrix $\mathrm{B}=(0,1)^\top$ and $\mathrm{q}\in\mathbb{R}^{2\times 1}$ is the remaining nonlinear part of the error system. For parameters r = 0.4, K = 500, d = 0.08, c = 0.8, T = 0.5, α = 0.001 and given desired state χ^*=18, we compute y^*=389.0704, u^*=25.57299. The matrix (20) has the form

$$A = \begin{pmatrix} 0.3712 & -2 \\ 0 & 0.8 \end{pmatrix}.$$

Choosing

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R = (1)$$

and solving the algebraic Riccati equation we obtain

$$P = \begin{pmatrix} 2.2697 & -1.6386 \\ -1.6386 & 4.6629 \end{pmatrix}.$$

Finally based on the Theorem 1 we compute the optimal control function $\tilde{u}$ for the error system

$$\tilde{u} = 1.5472z - 3.5980v = 1.5472x - 3.5980y + 1372.026\,,$$

hence the optimal control fiction u of the Acarus-Cheyletus system equals

$$u = \tilde{u} + u^* = 1.5472x - 3.5980y + 1397.599\,.$$

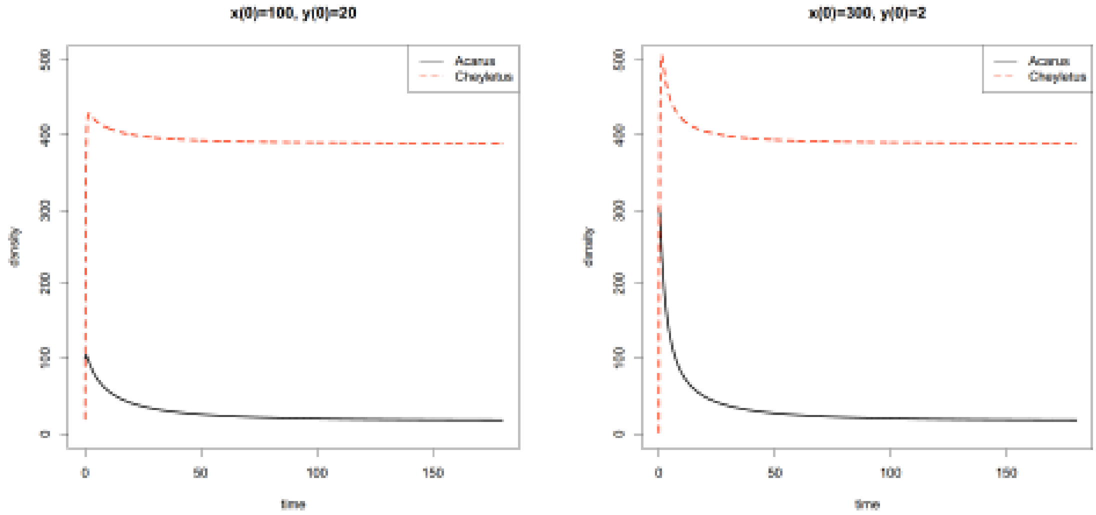

Figure 2. Density variations of the Acarus-Cheyletus model (14), (15) with control strategy

4. <u>Conclusion</u>

In this article we present a mathematical model for biological pest control. The control problem for nonlinear system is investigated in order to formulate the optimal control strategy by only introducing natural enemies. As an application we analyze the predator-pest system of two mites *Acarus siro* and *Cheyletus eruditus*. We derive the linear control u, that is a sum of the *feedforward control* function $u*$ and *feedback control* function $\tilde{u}$.. As the Figure 2 shows, the control u drives the trajectories of the Acarus-Cheyletus system from initial state after the massive augmentation of the Cheyletus population to the desired state $\chi*$ below the pest damage level.

References

[1] ANITA, S. (1997). Optimal harvesting for a nonlinear age-dependent population dynamics. J. Math. Analysis Appl., 226:6-22.

[2] BERNSTEIN, D. S. (1993). Nonquadratic cost and nonlinear feedback control. Int. J. Robust Nonlinear Control, 3:211–229. doi: 10.1002/rnc.4590030303.

[3] BODINE, E. N., GROSS, L. J., LENHART, S. (2008). Optimal control applied to a model for species augmentation. Math. Biosci. Eng., 5(4):669-80.

[4] BOUKAL, D. S., BEREC, L. (2009). Modelling mate-finding Allee effects and populations dynamics, with applications in pest control. Population Ecology 51:445-458. doi:10.1007/s10144-009-0154-4.

[5] BRYSON, A. E., Jr. (1996). Optimal control – 1950 to 1985. Control systems, IEEE, 16(3):26-33.

[6] KERN, D. L., LENHART, S., MILLER, R., YONG, J. (2007). Optimal control applied to native-invasive population dynamics. J. of Biological Dynamics, 1(4):413-426.

[7] KIRK, D. E. Optimal control theory: an introduction. Mineola, N.Y.: Dover Publications, 2004, 452 s. ISBN 0-486-43484-2.

[8] LUO, Z. (2007). Optimal harvesting for a population dynamics problem with age-structure and diffusion. J. Appl. Math. and Computing, 25(1-2):35-50.

[9] MURRAY, J. D. Mathematical Biology: I. An Introduction. Interdiscip. Appl. Math., Springer-Verlag, NY, 2002, 551 s. ISBN 0-387-95223-3.

[10] PEKÁR, S., ŽĎÁRKOVÁ, E. (2004). A model of the biological control of Acarus siro by Cheyletus eruditus (Acari:Acaridae, Cheyletidae) on grain. J. Pest Sci., 77:1-10.

[11] R Development Core Team (2008). An Introduction to R: Notes on R, A Programming Environment for Data Analysis and Graphics [electronic edition]. Accessible from: http://cran.r-project.org/doc/manuals/R-intro.html

[12] RUNGE, M. C., JOHNSON, F. A. (2002). The importance of functional form in optimal control solutions of problems in population dynamics. Ecology, 83(5):1357-1371.

[13] SHASTRI, Y., DIWEKAR, U. (2006). Sustainable ecosystem management using optimal control theory: Part 1 (deterministic systems). J. of Theoretical Biology, 241:506-521.

[14] VINCENT, T. L. (1975). Pest management programs via optimal control theory. Biometrics, 31(1):1-10.